Introduction to Abstract Algebra
with Applications to Social Systems

Women in Mathematics
SWIM
Summer Workshop in Mathematics
FOR HIGH SCHOOL STUDENTS

Princeton

Course II
Lecture
Notes
3 of 7

Princeton SWIM 2010

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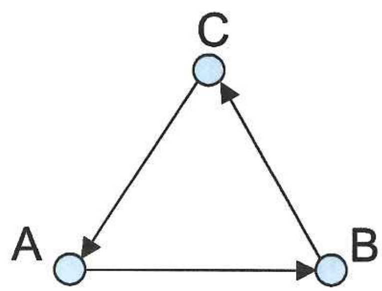
Communication Networks

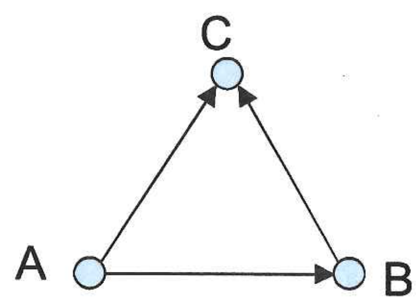
Dominance Relations

Dominance Relation: For each pair i, j, with $i \neq j$, either $A_i \rightarrow A_j$ or $A_j \rightarrow A_i$, but not both; that is, in every pair of individuals, there is exactly one who is dominant.

Tournaments



$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



$$D = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

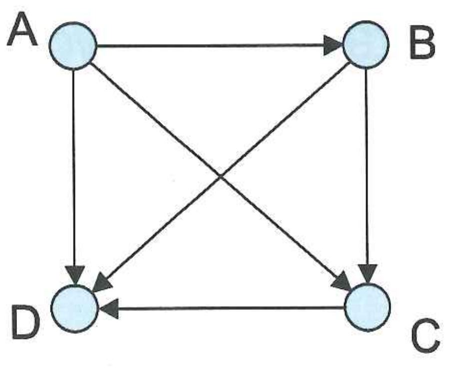
NOT Symmetric

Communication Networks

One-stage vs. Two-stage Communication

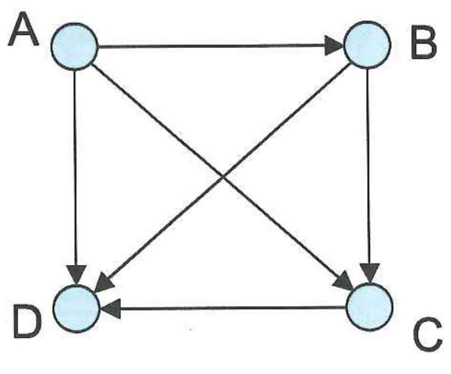
One-Stage

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



Two-Stage

$$C^2 = CC = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



Power

Dominance Matrices

One-Stage

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Two-Stage

$$D^2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Power: the total number of one-stage and two-stage dominances that an individual can exert. The power of individual A_i is the sum of the entries in the *i*th row of the matrix

$$S = D + D^2$$

Power

Example - Athletic Contest

The results of a round-robin athletic contest are shown below. Using the power definition above, rank the four teams in terms of their athletic dominance.

Team A beats teams B and D.
Team B beats team C.
Team C beats team A.
Team D beats teams C and B.

Power

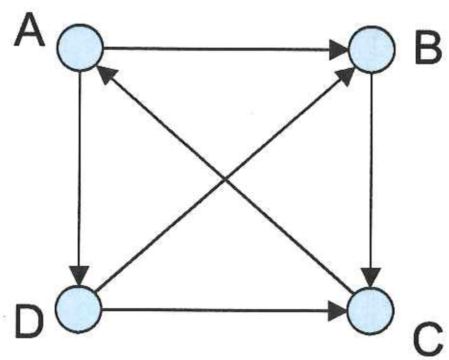
Example - Athletic Contest

Team A beats teams B and D.

Team B beats team C.

Team C beats team A.

Team D beats teams C and B.



$$D = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Power

Example - Athletic Contest

One-Stage

$$D = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$S = D + D^2 = \begin{pmatrix} 0 & 2 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

Row Sum
5
2
3
4

Two-Stage

$$D^2 = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Ranking:

1. Team A
2. Team D
3. Team C
4. Team B

Markov Chains

Definition

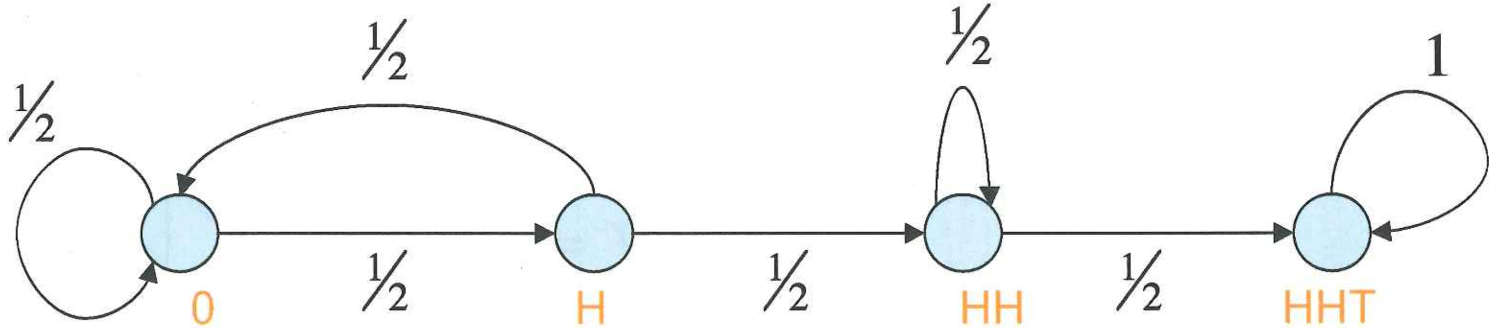
A **Markov process** is a stochastic (random) process with the following property:

The probability of any future behavior of the process depends only on the current state, not on its past behavior. (e.g., Markov property)

Markov Chains

Example - Flipping Coins

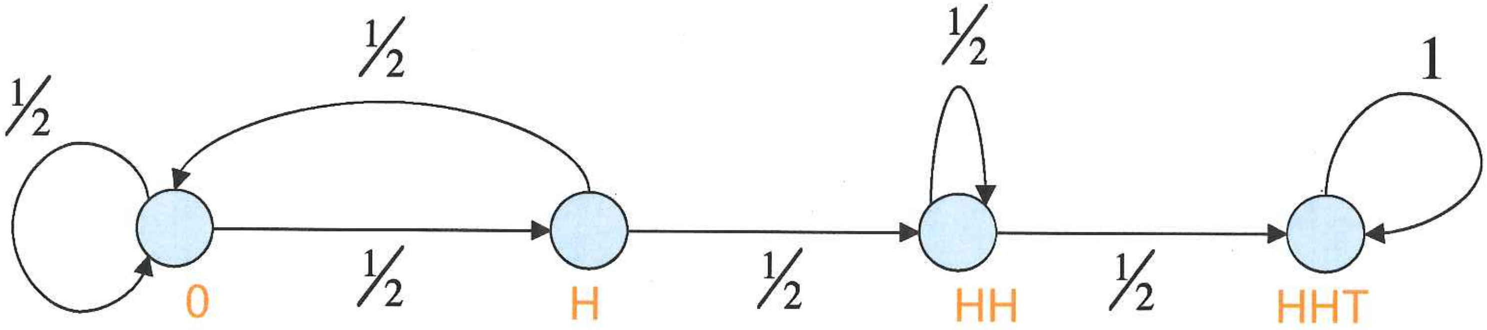
You are going to successively flip a quarter until the pattern HHT appears.



Markov Chains

Example - Flipping Coins

State Diagram



Transition Probability Matrix

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Markov Chains

Example - Flipping Coins

Question: On average, how many flips will be required until the pattern HHT appears?

Average number of flips required:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

We know that $v_4 = 0$. (Why?)

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{pmatrix}$$

Markov Chains

Example - Flipping Coins

Question: On average, how many flips will be required until the pattern HHT appears?

$$v = 1 + \beta v$$

$$v = 1 + \beta v = 1 + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 1 + \begin{pmatrix} \frac{1}{2}v_1 + \frac{1}{2}v_2 \\ \frac{1}{2}v_1 + \frac{1}{2}v_3 \\ \frac{1}{2}v_3 \end{pmatrix}$$

Markov Chains

Example - Flipping Coins

Question: On average, how many flips will be required until the pattern HHT appears?

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 + \tfrac{1}{2} v_1 + \tfrac{1}{2} v_2 \\ 1 + \tfrac{1}{2} v_1 + \tfrac{1}{2} v_3 \\ 1 + \tfrac{1}{2} v_3 \end{pmatrix}$$

$$v_1 = 1 + \tfrac{1}{2} v_1 + \tfrac{1}{2} v_2$$
$$v_2 = 1 + \tfrac{1}{2} v_1 + \tfrac{1}{2} v_3$$
$$v_3 = 1 + \tfrac{1}{2} v_3$$

Solve this system of equations

Markov Chains

Example - Flipping Coins

Question: On average, how many flips will be required until the pattern HHT appears?

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2} v_1 + \frac{1}{2} v_2 \\ 1 + \frac{1}{2} v_1 + \frac{1}{2} v_3 \\ 1 + \frac{1}{2} v_3 \end{pmatrix}$$

$$v_1 = 1 + \tfrac{1}{2} v_1 + \tfrac{1}{2} v_2$$
$$v_2 = 1 + \tfrac{1}{2} v_1 + \tfrac{1}{2} v_3$$
$$v_3 = 1 + \tfrac{1}{2} v_3$$

$\Rightarrow$

$$v_1 = 8$$
$$v_2 = 6$$
$$v_3 = 2$$
$$v_4 = 0$$

Markov Chains

Example - Flipping Coins

Question: In the long run, what fraction of time is spent in each state, no matter in which state the chain began at time 0?

THTHHTTHTHTHTHTHHTHTHHTTTHTHTHTHTHH

$$\pi T = \pi$$

Stationary Distribution